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On the correct expansion of a Green function into a set of eigenfunctions connected with a non-Hermitian eigenvalue problem considered by Morse

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Abstract. It is shown that the analysis of Morse of a non-Hermitian eigenvalue problem, connected with a string with non-rigid supports, leads to an erroneous expansion of the Green function into a set of related eigenfunctions. The correct expansion is derived.

1. Introduction

The following non-Hermitian eigenvalue problem was considered by Morse and Feshbach (1953): 'We suppose that a string of length l is under tension T and is supported by a rigid support at $x = 0$ and a non-rigid support at $x = l$. This latter support has enough longitudinal strength to support the tension T, but it yields a little to transverse force imparted to it by the string. Suppose this yielding involves both friction and stiffness of the support for sidewise motion, so that the relation between the transverse force transmitted by the string, which is $-T(\partial y/\partial x)_l$, is equal to R_s times the transverse velocity of the support, $(\partial y/\partial t)_l$, plus K_s times the displacement of the support $y(l)$:

$$-T\partial y/\partial x = R_s\partial y/\partial t + K_s y \qquad \text{at } x = l, \tag{1.1}$$

$$y = 0 \qquad \text{at } x = 0. \tag{1.2}$$

If we desire to compute the free vibrations of this system directly, we find the eigenfunction solutions of the wave equation

$$(\partial^2/\partial x^2 - c^{-2}\partial^2/\partial t^2)y = 0,$$

$$y = [A \sin(kx) + B \cos(kx)] \exp(-\mathrm{i}kct), \tag{1.3}$$

which have to satisfy the boundary conditions (1.1) and (1.2). They require that B be zero, so that

$$y_n = \sin(k_n x) \exp(-\mathrm{i}k_n ct), \tag{1.4}$$

where

$$-Tk_n \cos(k_n l) = -\mathrm{i}k_n cR_s \sin(k_n l) + K_s \sin(k_n l) \tag{1.5}$$

or

$$\tan(k_n l) = -Tk_n/(K_s - ik_n cR_s). \tag{1.6}$$

The roots k_n of this transcendental equation are the eigenvalues of the system.'

Suppose we want to determine the Green function G of the string, originally at rest in equilibrium, namely

$$(\partial^2/\partial x^2 - c^{-2}\partial^2/\partial t^2)G(x, x_0, t) = \delta(x - x_0)\delta(t), \tag{1.7}$$

satisfying the boundary and initial conditions

$$G(x, x_0, t) = 0, \qquad x = 0, \quad t \geq 0, \quad 0 \leq x_0 \leq l, \tag{1.8}$$

$$(T\partial/\partial x + R_s\partial/\partial t + K_s)G(x, x_0, t) = 0, \qquad x = l, \quad t \geq 0, \quad 0 \leq x_0 \leq l, \tag{1.9}$$

$$G(x, x_0, t) = 0, \qquad t \leq 0, \quad 0 \leq x \leq l, \quad 0 \leq x_0 \leq l, \tag{1.10}$$

$$(\partial/\partial t)G(x, x_0, t) = 0, \qquad t \leq 0, \quad 0 \leq x \leq l, \quad 0 \leq x_0 \leq l. \tag{1.11}$$

We expect that G can be expanded into the set of eigenfunctions $\{\sin(k_n x) \exp(\pm ick_n t)\}$ and consider to this end the properties of the functions

$$\psi_n(x) = \sin(k_n x). \tag{1.12}$$

These functions are solutions of the equation

$$(\mathrm{d}^2/\mathrm{d}x^2 + k_n^2)\psi_n(x) = 0, \tag{1.13}$$

and therefore, taking two different solutions ψ_1 and ψ_2 of (11), with numbers k_i satisfying (1.5), integration by parts leads to

$$-icR_sT^{-1}(k_1 - k_2)\psi_1(l)\psi_2(l) = (k_1^2 - k_2^2)\int_0^l \psi_1(x)\psi_2(x)\,\mathrm{d}x. \tag{1.14}$$

The RHS of (1.14) is *not* zero if $k_1 \neq k_2$, which shows that the eigenfunctions $\{\psi_n\}$ are not orthogonal. Moreover, it can be shown that the roots of the equation (1.5) have a non-vanishing imaginary part (see the Appendix). As every Hermitian eigenvalue problem leads to real eigenvalues and orthogonal sets of eigenfunctions, the results obtained above show that equations (1.5) and (1.3) define a non-Hermitian eigenvalue problem.

The non-Hermiticity of the eigenvalue problem set by equations (1.5) and (1.3) prevents us using the Sturm–Liouville theory, which is a special case of the theory of Hermitian operators. If equations (1.5) and (1.3) defines a Sturm–Liouville problem, the completeness and orthogonality of the set of eigenfunctions $\{\psi_n(x)\}$, namely

$$\delta(x - x_0) = \sum_n \psi_n(x)\psi_n^*(x_0), \tag{1.15}$$

show that the function

$$G(x, x_0) = \frac{1}{2\pi}\int_{-\infty+i\alpha}^{\infty+i\alpha} \exp(-ickt)\sum_n \frac{\psi_n(x)\psi_n^*(x_0)}{k^2 - k_n^2}\,\mathrm{d}k, \tag{1.16}$$

where α denotes an arbitrary positive number, satisfies equations (1.7)–(1.11).

The calculus of residues leads to

$$G(x, x_0, t) = \sum_n k_n^{-1}\psi_n(x)\psi_n^*(x_0)\sin(\omega_n t), \qquad \omega_n = ck_n, \quad t > 0. \tag{1.17}$$

We will derive in this paper a completeness relation for the set of functions $\{\sin(k_n x)\}$, where the numbers k_n are the roots of the equation (1.5). The desired completeness relation leads to an expansion of G in the set of modes $\{\psi_n\}$ similar to the expansion (1.17) for the case when the modes $\{\psi_n\}$ are the solutions of a Sturm–Liouville problem. We will first consider the treatment of the problem given by Morse and Feshbach (1953).

2. Morse solution of the problem

The difficulties arising from the non-Hermitian character of this problem were also noticed by Morse and Feshbach. In order to overcome these difficulties Morse and Feshbach proposed the following ingenious procedure.

First compute the Green function for a force of unit amplitude and frequency $(2\pi)^{-1}\omega$ applied at $x = x_0$, which leads to

$$(\mathrm{d}^2/\mathrm{d}x^2 + k^2)G(x, x_0) = -T^{-1}\delta(x - x_0), \qquad \omega = ck, \tag{2.1}$$

and

$$G(0, x_0) = 0, \qquad T(\partial/\partial x)G(x, x_0) = (\mathrm{i}kcR_s - K_s)G(x, x_0), \qquad x = l. \tag{2.2}$$

Equations (2.1) and (2.2) define an ordinary Sturm–Liouville problem because k is a *fixed* number. Standard techniques (Courant and Hilbert 1966, Morse and Feshbach 1953) lead to

$$G(x, x_0, k) = \frac{2l}{T}\sum_{n=0}^{\infty} \frac{2\pi\beta_n(k)}{2\pi\beta_n(k) - \sin(2\pi\beta_n(k))} \frac{\sin(\pi\beta_n(k)l^{-1}x_0)\sin(\pi\beta_n(k)l^{-1}x)}{(kl)^2 - (\pi\beta_n(k))^2}, \tag{2.3}$$

where $\beta_n(k)$ is the nth root of

$$\tan(\pi\beta_n) = kT/(\mathrm{i}kcR_s - K_s). \tag{2.4}$$

The function $g(x, x_0, t)$, satisfying

$$(\partial^2/\partial x^2 - c^{-2}\partial^2/\partial t^2)g(x, x_0, t) = \delta(x - x_0)\delta(t), \tag{2.5}$$

is the inverse Laplace transform of $G(x, x_0 k)$:

$$g(x, x_0, t) = \frac{c}{\pi l T}\sum_{n=0}^{\infty}\int_{-\infty+\mathrm{i}\alpha}^{\infty+\mathrm{i}\alpha} \frac{2\pi\beta_n(k)}{2\pi\beta_n(k) - \sin(2\pi\beta_n(k))} \times \frac{\sin(\pi\beta_n(k)l^{-1}x_0)\sin(\pi\beta_n(k)l^{-1}x)}{k^2 - \pi^2\beta_n^2(k)l^{-2}} \exp(-\mathrm{i}ckt)\,\mathrm{d}k. \tag{2.6}$$

The real number α has to be chosen in such a way that the singularities of the integrand are situated above the line $y = \mathrm{i}\alpha$. Morse evaluates the integral (2.6) using the theorem of residues. However, he only considers the residues arising from the zeros of

$$k - \beta_n(k)l^{-1} = 0 \qquad \text{or} \qquad k + \beta_n(k)l^{-1} = 0 \tag{2.7}$$

(which, considering equation (2.4), lead to the transcendental equation (1.6)), and neglects the possible roots of $2\pi\beta_n - \sin(2\pi\beta_n) = 0$. Moreover, the residue at a zero $k = k_n$ of equation (2.6) is *not*

$$\frac{\sin(k_n x)\sin(k_n x_0)}{2k_n l - \sin(2k_n l)} \exp(-\mathrm{i}ck_n t),$$

as was used by Morse and Feshbach, but rather

$$\frac{2c\mathrm{i}\sin(k_n x)\sin(k_n x_0)\exp(-\mathrm{i}ck_n t)}{T[2k_n l-\sin(2k_n l)](\partial/\partial k_n)(k-\pi\beta_n(k)l^{-1})}. \tag{2.8}$$

We will derive the correct expansion of the function $g(x, x_0, t)$ in the eigenfunctions (1.12) in the next section.

3. The expansion of the function $g(x, x_0, t)$ in natural modes

We will use the following theorem, rigorously derived by Geppert (1924), but which can be traced back to Cauchy (1827a, b, c) (see also Picard 1905):

Theorem. Let a, x_0 and x_1 denote real numbers and x and μ real variables such that

$$x_0<\mu<x_1, \qquad x_0<x<x_1. \tag{3.1}$$

Suppose that the function $f(\mu)$ is of bounded variation on the interval $x_0<\mu<x_1$ and that $\phi(k)$ and $\psi(k)$ denote polynomials of the same degree of the complex variable k. If the infinite set of real numbers $\{c_n\}$, $n=0, 1, 2, \ldots$, with $c_n\to\infty$ if $n\to\infty$, denote the radii of a set of circles in the complex k plane whose centres are situated at the origin of the real and imaginary axes and pass between two zeros of

$$L(k)=\exp(ak)\phi(k)-\exp(-ak)\psi(k), \tag{3.2}$$

we have

$$2\pi\mathrm{i}f(x)=\lim_{n\to\infty}\oint_{|k|=c_n}\frac{\exp(ak)\phi(k)\int_{x_0}^{x_1}\exp[(k(x-\mu)]f(\mu)\,\mathrm{d}\mu}{L(k)}\,\mathrm{d}k \qquad \text{if } 2a\geqslant x_1-x_0. \tag{3.3}$$

Let us change the variable k occurring in equation (1.5) into $\mathrm{i}k'$. This leads to

$$-\mathrm{i}Tk'\cosh(k'l)=\mathrm{i}k'cR_s\sinh(k'l)+\mathrm{i}K_s\sinh(k'l), \tag{3.4}$$

as well as to the set of eigenfunctions $\{\sinh(k'_n x)\}$. Let us take

$$x_1=l, \qquad a=l, \qquad \phi(k')=\mathrm{i}Tk'+\mathrm{i}k'cR_s+\mathrm{i}K_s,$$
$$x_0=-l, \qquad \psi(k')=-\mathrm{i}Tk'+\mathrm{i}k'cR_s+\mathrm{i}K_s. \tag{3.5}$$

Assuming, as Morse did, that the roots of equation (3.4) are simple (see Nussenzveig (1972) for a discussion of a similar problem arising in S-matrix theory), equations (3.2)–(3.5) and the theorem of residues lead to

$$f(\pm x)=\sum_n\frac{\exp(lk'_n)\phi(k'_n)\int_{-l}^{+l}\exp k'_n(x\mp\mu)f(\mu)\,\mathrm{d}\mu}{\dot{L}(k'_n)}, \tag{3.6}$$

where

$$\dot{L}(k'_n)=(\mathrm{d}/\mathrm{d}k)L(k)|_{k=k_n} \tag{3.7}$$

Equation (3.6) shows that, if $f(x)$ is identically zero on the interval $-l\leqslant\mu\leqslant 0$, we obtain

$$f(x)=-2\sum_n\frac{\exp(lk'_n)\phi(k'_n)\exp(k'_n x)\int_0^l\sinh(k'_n\mu)f(\mu)\,\mathrm{d}\mu}{\dot{L}(k'_n)}. \tag{3.8}$$

Therefore

$$\delta(x-\mu) = -2\sum_n \frac{\exp(lk'_n)\phi(k'_n)\exp(k'_nx)\sinh(k'_n\mu)}{\dot{L}(k'_n)}, \qquad 0<\mu<l, \quad 0<x<l, \tag{3.9}$$

which is the desired completeness relation for the set of eigenmodes $\{\sin k_nx\}$. Inserting

$$G(x, x_0, t) = -\frac{\mathrm{i}}{\pi}\int_{-\infty+\mathrm{i}\alpha}^{\infty+\mathrm{i}\alpha} \exp(-\mathrm{i}kct)\sum_n \frac{1}{\dot{L}(-\mathrm{i}k_n)(k^2-k_n^2)}\exp(-\mathrm{i}lk_n) \times\phi(-\mathrm{i}k_n)\exp(-\mathrm{i}k_nx_0)\sin(k_nx)\,\mathrm{d}k \tag{3.10}$$

into the wave equation shows that, using the completeness relation (3.9), $G(x, x_0, t)$ is a solution of equation (1.7) satisfying the boundary and initial conditions (1.8)–(1.11), if the real number α is chosen such that the singularities of the integrand of (3.10) are situated below the line $y = \mathrm{i}\alpha$ in the complex k plane. The theorem of residues leads to

$$\begin{aligned} G(x, x_0, t) &= \mathrm{i}\sum_n \sin(ck_nt)\frac{1}{\dot{L}(-\mathrm{i}k_n)k_n} \\ &\quad\times\exp(-\mathrm{i}lk_n)\phi(-\mathrm{i}k_n)\exp(-\mathrm{i}k_nx_0)\sin(k_nx), \qquad t>0, \\ G(x, x_0, t) &= 0, \qquad t<0. \end{aligned} \tag{3.11}$$

4. Discussion

The expansion (3.11) of the Green function in terms of the natural modes of the system was the aim of Morse's analysis. A related problem, connected with the completeness of the natural modes of an embedded dielectric sphere, as well as a survey of the occurrence of natural modes in physics, has been recently considered by Hoenders (1978). This completeness problem was analysed with the methods of Cauchy and Geppert.

Though the expansion (3.11) must somehow be contained in his integral (2.6), the author fails to derive (3.11) from (2.6) for two reasons:

(1) It is difficult to evaluate the possible residues generated by $2\pi\beta_n(k) - \sin(2\pi\beta_n(k)) = 0$.

(2) The evaluation of the integral (2.6) by the methods of contour integration leads to contributions from the poles of the integrand but also to contributions generated by the occurrence of branch points: the function $(kl)^2 - (\pi\beta_n(k))^2$ contains branch points because (see equation (2.4))

$$\pi\beta_n(k) = \tan^{-1} f(k) = \frac{1}{2\mathrm{i}}\ln\left(\frac{1+\mathrm{i}f(k)}{1-\mathrm{i}f(k)}\right), \tag{4.1}$$

with

$$f(k) = kT/(\mathrm{i}kCR_s - K_s). \tag{4.2}$$

These contributions lead to an additional difficulty for the evaluation of the integral (2.6).

All these difficulties can be avoided if this boundary and initial value problem is analysed by the method of Cauchy (1827a, b, c), which was extended and made more rigorous by Geppert (1924). This point of view is supported by the analysis of a related problem connected with the completeness of the natural modes of a dielectric sphere embedded in an infinite dielectric medium by Hoenders (1978). The completeness problem is solved by the Cauchy method, and a survey of the occurrence of natural modes in physics in the Introduction of this paper indicates that this method 'naturally' solves different kinds of boundary and initial value problems arising in physics.

Appendix

The asymptotic behaviour of the roots k_n can be determined in a more general way than that proposed by Morse and Feshbach (1953). Equation (1.5),

$$kT[\cos(kl) - \mathrm{i}cR_s T^{-1}\sin(kl)] = -K_s \sin(kl), \tag{A1}$$

leads to

$$\cos(kl+\phi) = -[K_s \cos\phi \sin(kl)]/Tk \tag{A2}$$

if

$$\tan\phi = \mathrm{i}cR_s T^{-1}. \tag{A3}$$

For large values of $|k_n|$ we expect that $k_n \simeq (n+\frac{1}{2})\pi l^{-1} - \phi$, and therefore insert

$$k_n = (n+\tfrac{1}{2})\pi l^{-1} - \phi + \epsilon \tag{A4}$$

into (A2). This leads to

$$(-1)^{n+1}\sin\epsilon = -\frac{K_s\cos\phi\sin[(n+\frac{1}{2})\pi - \phi + \epsilon]}{[(n+\frac{1}{2})\pi l^{-1} - \phi + \epsilon]T} \tag{A5}$$

and

$$\epsilon = (-1)^n \frac{K_s\cos\phi\sin[(n+\frac{1}{2})\pi - \phi]}{(n+\frac{1}{2})\pi l^{-1} - \phi} + \mathrm{O}\left(\frac{1}{n^2}\right). \tag{A6}$$

Combination of equations (A4) and (A6) leads to

$$k_n = (n+\tfrac{1}{2})\pi l^{-1} - \phi + (-1)^n \frac{K_s\cos\phi\sin[(n+\frac{1}{2})\pi - \phi]}{(n+\frac{1}{2})\pi l^{-1} - \phi} + \mathrm{O}\left(\frac{1}{n^2}\right). \tag{A7}$$

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